

CALCULUS II TUTORIAL: SOLUTIONS (6)

① Put $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

Then $y' = \frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots$

$\Rightarrow y' + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \dots$

We require $y' + y = x$. $\boxed{+ (na_n + a_{n-1})x^{n-1} + \dots}$

Comparing different powers of x gives

$a_1 + a_0 = 0$ [x^0], $2a_2 + a_1 = 1$ [x^1],

$3a_3 + a_2 = 0$ [x^2], $na_n + a_{n-1} = 0$ [x^{n-1}].

Also $y(0) = 1 \Rightarrow a_0 = 1$. We therefore have

$a_1 = -a_0 = -1$, $a_2 = \frac{1 - a_1}{2} = \frac{2}{2}$, $a_3 = -\frac{a_2}{3} = -\frac{2}{3 \cdot 2}$,

$a_n = -\frac{a_{n-1}}{n} = +\frac{a_{n-2}}{n(n-1)} = -\frac{a_{n-3}}{n(n-1)(n-2)} = \dots = (-1)^n \frac{2}{n!}$

Solution can then be written as

$y = 1 - x + 2 \left[\frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^n}{n!} + \dots \right]$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 1 - x + \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n!}$

$\Rightarrow y = 1 - x + 2(e^{-x} - 1 + x) = 2e^{-x} - 1 + x$

Many types of differential equations can be solved using this power series technique.

REMINDER FOR TUTORS

Alternating Series Estimation Theorem:

If $S_n = u_1 - u_2 + \dots + (-1)^{n+1}u_n$ with $u_n > 0$,

$u_n \geq u_{n+1}$ for $n \geq N$ for some N , $u_n \rightarrow 0$, then $S_n \rightarrow L$,

$L - S_n$ has same sign as u_{n+1} term and $|L - S_n| < u_{n+1}$.

Remainder Estimation Theorem:

$f(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$
for some c between a and x .

②

$$(a) f(x) = x^3 - 2x + 4 \Rightarrow f'(x) = 3x^2 - 2 \Rightarrow f''(x) = 6x \Rightarrow f'''(x) = 6$$

$$\text{Hence } f(2) = 8, f'(2) = 10, f''(2) = 12, f'''(2) = 6$$

$$\Rightarrow f(x) = f(2) + f'(2)(x-2) + \frac{1}{2!} f''(2)(x-2)^2 + \frac{1}{3!} f'''(2)(x-2)^3$$

$$= 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$$

$$(b) f(x) = \frac{1}{x^2} \Rightarrow f'(x) = -\frac{2}{x^3} \Rightarrow f''(x) = \frac{2 \cdot 3}{x^4} \Rightarrow f^{(n)}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}}$$

$$\text{Hence } f^{(n)}(1) = (-1)^n (n+1) \Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(1)(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x-1)^n}{n!}$$

$$(c) f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \Rightarrow f^{(n)}(2) = e^2$$

$$\text{Hence } f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(2)(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$$

③

$$(a) f(x) = \ln(\cos x) \Rightarrow f'(x) = \frac{-\sin x}{\cos x} = -\tan x \Rightarrow f''(x) = -\sec^2 x$$

$$\text{Hence } f(0) = \log 1 = 0, f'(0) = 0, f''(0) = -1$$

$$\Rightarrow P_1(x) = f(0) + x f'(0) = 0, P_2(x) = P_1(x) + \frac{1}{2} x^2 f''(0) = -\frac{1}{2} x^2$$

$$(b) f(x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow f'(x) = \frac{-(-2x)}{2(1-x^2)^{3/2}} = \frac{x}{(1-x^2)^{3/2}} \Rightarrow f'(0) = 0$$

$$\Rightarrow f''(x) = \frac{1}{(1-x^2)^{3/2}} - \frac{3}{2} \frac{x(-2x)}{(1-x^2)^{5/2}} = \frac{1}{(1-x^2)^{3/2}} + \frac{3x^2}{(1-x^2)^{5/2}} \Rightarrow f''(0) = 1$$

$$\text{Hence } P_1(x) = f(0) + x f'(0) = 1, P_2(x) = P_1(x) + \frac{1}{2} x^2 f''(0) = 1 + \frac{1}{2} x^2$$

④

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \text{ (from notes) . Using substitution}$$

$$(a) \cos \sqrt{x+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(x+1)^n}{2^n} = 1 - \frac{(x+1)}{2} + \frac{(x+1)^2}{24} - \dots$$

$$(b) \cos \left(\frac{x^{3/2}}{\sqrt{2}} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{x^{3/2}}{\sqrt{2}} \right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{2^n (2n)!} = 1 - \frac{x^3}{4} + \frac{x^6}{96} - \dots$$

$$(c) \sin^2 x = \frac{1}{2} (1 - \cos 2x) = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} = x^2 - \frac{1}{3} x^4 + \dots$$

$$\textcircled{5} e^x = 1 + x + \frac{x^2}{2!} + R_3(x) \text{ with } R_3(x) = \frac{x^3}{3!} \cdot \frac{d^3(e^x)}{dx^3} \Big|_{x=c} = \frac{e^c x^3}{6}$$

where $R_3(x)$ is remainder of order 3 and $0 < c < x$.

$$|\text{Error}| \text{ for } |x| < 0.1 \text{ is less than } \frac{e^{0.1} (0.1)^3}{6} < \frac{3^{0.1} (0.1)^3}{6} = 1.9 \times 10^{-4}$$

For $x < 0$, $e^x = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots$ (i.e. alternating series)

$$\text{In this case, } |\text{error}| \text{ less than } \frac{1}{6} (0.1)^3 = 1.7 \times 10^{-4}$$

(by Alternating Series Estimation Thm) which is slightly less.

$$\textcircled{6} \text{ (a) } \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \text{ (from lectures)}$$

ASET $\Rightarrow$ |error| in using $\ln(1+x) = x$ is less than $\frac{1}{2}x^2$.

$$\text{Since } \left| \frac{\text{error}}{x} \right| = \frac{1}{2}|x| < \frac{1}{100}, \text{ we need } |x| < 0.02$$

(b)

ASET $\Rightarrow$ error has same sign as first neglected term.

$$\text{Since } \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \text{ (from lectures),}$$

$$\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \dots \text{ Hence } 1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1$$

$\textcircled{7}$

$$\text{(a) } \left(1 + \frac{x}{2}\right)^{-2} = 1 + (-2)\left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2 \cdot 1}\left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3 \cdot 2 \cdot 1}\left(\frac{x}{2}\right)^3 + \dots$$

$$= 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \dots$$

$$\text{(b) } \left(1 + \frac{1}{x}\right)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)\left(\frac{1}{x}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2 \cdot 1}\left(\frac{1}{x}\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3 \cdot 2 \cdot 1}\left(\frac{1}{x}\right)^3 + \dots$$

$$= 1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3} - \dots$$

(c)

$$\left(1 + x^2\right)^{-\frac{1}{3}} = 1 + \left(-\frac{1}{3}\right)x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2 \cdot 1}(x^2)^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3 \cdot 2 \cdot 1}(x^2)^3 + \dots$$

$$= 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 + \dots$$

$\textcircled{8}$

$$\text{(a) } \int_0^{0.1} \sqrt{1+x^4} dx = \int_0^{0.1} \left\{ 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \dots \right\} dx \approx \left[x + \frac{1}{10}x^5 - \frac{1}{72}x^9 \right]_0^{0.1}$$

$$= (0.1) + \frac{1}{10}(0.1)^5 - \frac{1}{72}(0.1)^9 \approx 0.100001$$

(b)

$\hookrightarrow$ error around $10^{-11} < 10^{-8}$

$$\frac{\ln(1+x^2)}{1-\cos x} = \frac{(x^2) - \frac{1}{2}(x^2)^2 + \dots}{1 - \left\{ 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots \right\}} = \frac{x^2 - \frac{1}{2}x^4 + \dots}{\frac{1}{2}x^2 - \frac{1}{24}x^4 + \dots} \rightarrow 2 \text{ as } x \rightarrow 0$$

where we've used series of $\ln(1+x)$ and $\cos x$.